(VOLUME) DENSITY PROPERTY OF A FAMILY OF COMPLEX MANIFOLDS INCLUDING THE KORAS-RUSSELL CUBIC

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ABSTRACT. We present modified versions of existing criteria for the density property and the volume density property of complex manifolds. We apply this methods to show the (volume) density property for a family of manifolds given by $x^2y = a(\bar{z}) + xb(\bar{z})$ with $\bar{z} = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ and volume form $\mathrm{d}x/x^2 \wedge \mathrm{d}z_0 \wedge \ldots \wedge \mathrm{d}z_n$. The key step is showing that in certain cases transitivity of the action of (volume preserving) holomorphic automorphisms implies the (volume) density property, and then giving sufficient conditions for the transitivity of this action. In particular, we show that the Koras-Russell Cubic Threefold $\{x^2y + x + z_0^2 + z_1^3 = 0\}$ has the density property and the volume density property.

1. Introduction

The density property and the volume density property is a property of Stein manifolds with a huge amount of applications in complex geometry in several variables. It was introduced by Varolin in [17]. The fact that $\mathbb{C}^n$ has the density property for $n \geq 2$ was already used by Andersén and Lempert in [1] where they showed that holomorphic automorphisms can be approximated by some special family of automorphisms (called shear automorphisms). Varolin realized that the main observation of Andersén and Lempert may be formalized and can be applied to more general complex manifolds and different problems. Rosay and Forstnerič contributed also a lot to this progress in [4]. This area of complex analysis in several variables is nowadays called Andersén-Lempert theory. The numerous applications of the density property are due to the Main Theorem of Andersén-Lempert theory which states that on manifolds with density property any local phase flow on a Runge domain can be approximated uniformly on compacts by global automorphisms. The analogous statement holds in the volume preserving case. For a deeper view into this topic we refer to the comprehensive texts [3, 7, 10].

Definition 1.1. Let X be a Stein manifold. If the Lie algebra $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X))$ generated by complete (= globally integrable) holomorphic vector fields $\mathrm{CVF}_{\mathrm{hol}}(X)$ on X is dense (in compact-open topology) in the Lie algebra of all holomorphic vector fields $\mathrm{VF}_{\mathrm{hol}}(X)$ on X then X has the density property.

Let X be a Stein manifold equipped with a holomorphic volume form ω. If the Lie algebra $\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X))$ generated by complete volume preserving (= vanishing ω-divergence) holomorphic vector fields $\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)$ on X is dense in the Lie algebra

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of all volume preserving holomorphic vector fields $\mathrm{VF}^{\omega}_{\mathrm{hol}}(X)$ on X then X has the volume density property.

Recall that a vector field ν is called volume preserving if the Lie derivative $L_\nu \omega$ vanishes where the Lie derivative is given by the formula $L_\nu = \mathrm{d}i_\nu + i_\nu \mathrm{d}$ and i_ν is the interior product of a form with ν.

In Section 2 we present a criterion that is showing (volume) density property. For the definition of (semi-)compatible pairs and (ω-)generating sets see Definitions 2.8, 2.10 and 2.4. For a vector field ν and a point $p \in X$ we denote by $\nu[p] \in \mathrm{T}_p X$ the tangent vector of ν at p.

Theorem 1.2. *(1) Let X be a Stein manifold such that the holomorphic automorphisms $\mathrm{Aut}_{\mathrm{hol}}(X)$ act transitively on X. If there are compatible pairs (ν_i, μ_i) such that there is a point $p \in X$ where the vectors $\mu_i[p]$ form a generating set of $\mathrm{T}_p X$ then X has the density property.*

(2) Let X be a Stein manifold with a holomorphic volume form ω such that the volume preserving holomorphic automorphisms $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)$ act transitively on X and $\mathrm{H}^{n-1}(X, \mathbb{C}) = 0$ (where $n = \dim X$). If there are semi-compatible pairs (ν_i, μ_i) of volume preserving vector fields such that there is a point $p \in X$ where the vectors $\nu_i[p] \wedge \mu_i[p]$ form a generating set of $\mathrm{T}_p X \wedge \mathrm{T}_p X$ then X has the volume density property.

Note that this criterion and its proof is very much inspired by the criteria in [5,8] for the algebraic (volume) density property (see Definition 5.3). Actually, e.g. for (1), the only difference is that instead of requiring the algebraic automorphisms to act transitively on X we require the holomorphic automorphisms to act transitively.

In Section 3 we investigate the transitivity of the action by (volume preserving) holomorphic automorphisms $\mathrm{Aut}_{\mathrm{hol}}(X)$ (resp. $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)$) where X is given by $x^2 y = a(\bar{z}) + x b(\bar{z})$ with $\bar{z} = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ for some $n \geq 0$, $\deg_{z_0}(a) \leq 2$ and $\deg_{z_0}(b) \leq 1$. We show that (after possibly reordering the z_i) the condition

(A) There is some $k \geq 0$ such that $\deg_{z_i}(a) \leq 2$ and $\deg_{z_i}(b) \leq 1$ for all $i \leq k$ and for all common zeroes $\bar{q} = (q_0, \ldots, q_n)$ of $a, \frac{\partial a}{\partial z_0}, \ldots, \frac{\partial a}{\partial z_k}$, we have $b(\bar{q}) \neq 0$, and there is some $j \leq k$ such that $\frac{\partial a}{\partial z_j}$ does not vanish along the curve $\{z_i = q_i \text{ for all } i \neq j\} \subset \mathbb{C}^{n+1}$.

is a sufficient condition for $\mathrm{Aut}_{\mathrm{hol}}(X)$ to act transitively on X. If additionally

(B) There is some $k \geq 0$ such that $\deg_{z_i}(a) \leq 2$ and $\deg_{z_i}(b) \leq 1$ for all $i \leq k$ and there is no $c \in \mathbb{C}^*$ for which the polynomials $\frac{\partial a}{\partial z_i} + c\frac{\partial b}{\partial z_i}$ for $i \leq k$ are all constant to zero.

holds, then also $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)$ acts transitively (Proposition 3.4).

In Section 4 we apply Theorem 1.2 to these kind of surfaces for $n > 0$. This leads to our Main Theorem.

Main Theorem. *Let $n \geq 0$ and $a, b \in \mathbb{C}[z_0, \ldots, z_n]$ such that $\deg_{z_0}(a) \leq 2$, $\deg_{z_0}(b) \leq 1$ and not both of $\deg_{z_0}(a)$ and $\deg_{z_0}(b)$ are equal to zero. Let $\bar{z} = (z_0, \ldots z_n)$. Then the hypersurface $X = \{x^2 y = a(\bar{z}) + x b(\bar{z})\}$ has the density property provided that the holomorphic automorphisms $\mathrm{Aut}_{\mathrm{hol}}(X)$ act transitively on X. In particular X has the density property if (A) holds or if $n = 0$.*

Moreover, if $\mathrm{H}^{n+1}(X, \mathbb{C}) = 0$ and the volume preserving holomorphic automorphisms $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)$ act transitively on X then X has the volume density property for *the volume form* $\omega = \mathrm{d}x/x^2 \wedge \mathrm{d}z_0 \wedge \ldots \wedge \mathrm{d}z_n$. *In particular the transitivity condition holds if (A) and (B) hold or if* $n = 0$.

The proof of the Main Theorem is finished in Section 5 where the case $n = 0$ is done by explicit calculations, not using the methods described in Section 2.

It is worth pointing out that the Main Theorem together with Corollary 3.5 implies that the Koras-Russell Cubic Threefold $C = \{x^2y + x + z_0^2 + z_1^3 = 0\}$ has the (volume) density property. The Threefold C is a famous example of a affine variety which is diffeomorphic to $\mathbb{R}^6$ but not algebraically isomorphic to $\mathbb{C}^3$, e.g. see [13]. As an affine algebraic variety C (in particular the algebraic automorphism group of C) is well understood, e.g. see [2]. For example it is know that the algebraic automorphisms does not act transitively on C. The density property implies that the situation in the holomorphic context is completely different. However, it is still unclear if C is biholomorphic to $\mathbb{C}^3$. Related to this question there is a conjecture of Tóth and Varolin. The conjecture [16] states that a manifold which has the density property and which is diffeomorphic to $\mathbb{C}^n$ is automatically biholomorphic to $\mathbb{C}^n$. If the conjecture holds then the Main theorem would imply that C is isomorphic to $\mathbb{C}^3$.

2. Proof of Theorem 1.2

Let X be a Stein manifold of dimension n, and let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X.

2.1. **Preliminaries.**

Let $\mathfrak{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules, and let $s_1, \ldots, s_N \in \mathfrak{F}(X)$ be global sections. The following lemmas are standard applications of sheaf theory.

Lemma 2.1. *Let $p \in X$, and let $\mu_p \subset \mathcal{O}_X(X)$ be corresponding ideal. If the elements $s_i + \mu_p\mathfrak{F}(X)$ span the vector space $\mathfrak{F}(X)/\mu_p\mathfrak{F}(X)$ then the localizations $(s_i)_p$ generate the stalk $\mathfrak{F}_p$.*

Proof. Let $\mathfrak{G}_p$ be the $(\mathcal{O}_X)_p$-submodule of $\mathfrak{F}_p$ generated by $(s_1)_p, \ldots, (s_N)_p$. The assumption implies that we have $(\mathfrak{G}_p + \mu_p\mathfrak{F}_p)/\mu_p\mathfrak{F}_p = \mathfrak{F}_p/\mu_p\mathfrak{F}_p$. Let $\mathfrak{L}_p = \mathfrak{F}_p/\mathfrak{G}_p$ then a short calculation using the isomorphism theorems for modules shows that we have $\mathfrak{L}_p/\mu_p\mathfrak{L}_p = 0$. The ring $(\mathcal{O}_X)_p$ is local, so by the Nakayama Lemma we may lift a basis of $\mathfrak{L}_p/\mu_p\mathfrak{L}_p$ to a generating set of $\mathfrak{L}_p$, and thus $\mathfrak{L}_p = 0$. This yields $\mathfrak{G}_p = \mathfrak{F}_p$ which shows the claim. □

Lemma 2.2. *If the elements $(s_i)_p$ generate the stalks $\mathfrak{F}_p$ for all points $p \in X$ then every global section $\nu \in \mathfrak{F}(X)$ is of the form $\sum f_i s_i$ for some global holomorphic functions $f_i \in \mathcal{O}_X(X)$.*

Proof. Consider the morphisms of sheaves $\varphi : \mathcal{O}_X^N \to \mathfrak{F}$ given by $(f_i) \mapsto \sum f_i s_i$. By assumption φ is surjective on the level of stalk. Therefore we get the following short exact sequence of coherent sheaves:

$$0 \to \ker\varphi \to \mathcal{O}_X^N \to \mathfrak{F} \to 0.$$

This yields the following long exact sequence:

$$\cdots \to \mathrm{H}^0(X, \mathcal{O}_X^N) \to \mathrm{H}^0(X, \mathfrak{F}) \to \mathrm{H}^1(X, \ker\varphi) \to \cdots.$$

By Theorem B of Cartan we have $\mathrm{H}^1(X, \ker\varphi) = 0$. Thus the left map is surjective, and therefore every global section $\nu \in \mathrm{H}^0(X, \mathfrak{F}) = \mathfrak{F}(X)$ is of the desired form. □

Recall that a domain $Y \subset X$ is called Runge if all holomorphic function on Y can be approximated uniformly on compacts $K \subset Y$ by global holomorphic functions on X.

Lemma 2.3. *Let $Y \subset X$ be a domain of X which is Runge and Stein. If the elements $(s_i)_p$ generate the stalks $\mathfrak{F}_p$ for all points $p \in Y$ then every global section $\nu \in \mathfrak{F}(X)$ can be uniformly approximated on compacts $K \subset Y$ by global sections of the form $\sum f_i s_i$ for some global holomorphic functions $f_i \in \mathcal{O}_X(X)$.*

Proof. Let $\nu|_Y \in \mathfrak{F}(Y)$ be the restriction of ν to Y. By Lemma 2.2 we have $\nu|_Y = \sum g_i s_i|_Y$ for some holomorphic functions $g_i \in \mathcal{O}_X(Y)$ on Y. Since Y is a Runge domain we may approximate the functions g_i by global functions $f_i \in \mathcal{O}_X(X)$ uniformly on compacts $K \subset Y$. Thus the global section ν can be approximated by sections $\sum f_i s_i$ uniformly on compacts $K \subset Y$. □

2.2. **Criterion for (volume) density property.** The following definition is due to [5], but adapted to the holomorphic case.

Definition 2.4. Let $p \in X$. A set $U \subset \mathrm{T}_p X$ is called generating set for $\mathrm{T}_p X$ if the orbit of U of the induced action of the stablilizer $\mathrm{Aut}_{\mathrm{hol}}(X)_p$ contains a basis of $\mathrm{T}_p X$.

If X has a volume form ω then a set $U \subset \mathrm{T}_p X \wedge \mathrm{T}_p X$ is called ω-generating set for $\mathrm{T}_p X \wedge \mathrm{T}_p X$ if the orbit of U of the induced action of the stabilizer $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)_p$ contains a basis of $\mathrm{T}_p X \wedge \mathrm{T}_p X$.

The next proposition a powerful criterion for the density property. It is a generalization of Theorem 2 in [5] and the proof is similar.

Proposition 2.5. *Let X be a Stein manifold such that $\mathrm{Aut}_{\mathrm{hol}}(X)$ acts transitively on X. Assume that there are complete vector fields $\nu_1, \ldots, \nu_N \in \mathrm{CVF}_{\mathrm{hol}}(X)$ which generate a submodule that is contained in the closure of $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X))$ and assume that there is a point $p \in X$ such that the tangent vectors $\nu_i[p] \in \mathrm{T}_p X$ are a generating set for the tangent space $\mathrm{T}_p X$. Then X has the density property.*

Proof. We may assume that the vectors $\nu_i[p]$ contain a basis of $\mathrm{T}_p X$. Indeed, the vectors $\nu_i[p]$ are a generating set of $\mathrm{T}_p X$. Thus after adding some pull backs of some vector fields ν_i by automorphisms in $\mathrm{Aut}_{\mathrm{hol}}(X)_p$ we get the desired basis of $\mathrm{T}_p X$. Let $A \subsetneq X$ be the analytic subset of points where the vectors $\nu_i[a]$ do not span the whole tangent space $\mathrm{T}_a X$.

Let $K \subset X$ be a compact set. After replacing K by its $\mathcal{O}_X$-convex hull we may assume that K is $\mathcal{O}_X$-convex. Let Y be a neighborhood of K which is Stein and Runge, and moreover such that the closure of Y is compact. See e.g. the beginning of the proof of Theorem 7 in [15] for the existence of such a Y.

After adding finitely many complete vector fields to $\nu_1, \ldots, \nu_N$ we get that $Y \cap A = \emptyset$. Indeed, since the closure of Y is compact, $Y \cap A$ is a finite union of irreducible analytic subsets. Let $A_0 \subset A$ be an irreducible component of maximal dimension. Pick any $a \in A_0$ and $\phi \in \mathrm{Aut}_{\mathrm{hol}}(X)$ such that $\phi(a) \in Y \setminus A$. Since the vectors $\nu_i[\phi(a)]$ span the tangent space $\mathrm{T}_{\phi(a)} X$ the vectors $(\phi^* \nu_i)[a]$ span the tangent space $\mathrm{T}_a X$. Thus after adding some of the pull backs to $\nu_1, \ldots, \nu_N$ the component $A_0 \cap Y$ is replaced by finitely many components of lower dimension. Repeating the same procedure, inductively we get after finitely steps a list of complete vector fields $\nu_1, \ldots, \nu_N$ such that $A \cap Y = \emptyset$.

Let $\mathfrak{F}$ be the coherent sheaf corresponding to the tangent bundle. The fact that the vectors $\nu_i[a]$ span $\mathrm{T}_a X$ for all $a \in Y$ translates to the fact the elements $\nu_i + \mu_a \mathfrak{F}$ span the vector space $\mathfrak{F}/\mu_a\mathfrak{F}$ for all $a \in Y$, where μ_a is the maximal ideal of a. Thus by Lemma 2.1 the assumption of Lemma 2.3 holds. Therefore ever vector field on X can be approximated uniformly on K by elements of the form $\sum f_i \nu_i$ for some regular functions $f_i \in \mathcal{O}_X(X)$. By assumption the submodule generated by $\nu_1, \ldots, \nu_N$ is contained in the closure of $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X))$ (note that this property still holds after enlarging the list $\nu_1, \ldots, \nu_N$ in the procedure above). Therefore every holomorphic vector field is in the closure of $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X))$. □

For the volume case the proof can be found in [11]. For completeness indicate a proof here. We introduce the following isomorphisms. The article [8] is a good reference for this methods. Note the reference is written only for the algebraic case, however all isomorphisms exist identically in the holomorphic case. Let ω be a holomorphic volume form. For $0 \le i \le n$ let $\mathcal{C}_i(X)$ be the vector space of holomorphic i-forms on X. Moreover let $\mathcal{Z}_i(X) \subset \mathcal{C}_i(X)$ be the vector space of closed i-forms, and let $\mathcal{B}_i(X) \subset \mathcal{Z}_i(X)$ be the vector space of exact i-forms. Then we have the isomorphism:

$$\Phi : \mathrm{VF}_{\mathrm{hol}}(X) \tilde{\to} \mathcal{C}_{n-1}(X), \quad \nu \mapsto i_v \omega,$$

and in the same spirit we also have the isomorphism Ψ induced by:

$$\Psi : \mathrm{VF}_{\mathrm{hol}}(X) \wedge \mathrm{VF}_{\mathrm{hol}}(X) \tilde{\to} \mathcal{C}_{n-2}(X), \quad \nu \wedge \mu \mapsto i_\nu i_\mu \omega.$$

By definition $\mathrm{VF}^{\omega}_{\mathrm{hol}}(X)$ consists of those vector fields ν such that $L_\nu \omega = \mathrm{d} i_\nu \omega = 0$ holds and thus the isomorphism Φ restricts to an isomorphisms

$$\Theta = \Phi|_{\mathrm{VF}^{\omega}_{\mathrm{hol}}(X)} : \mathrm{VF}^{\omega}_{\mathrm{hol}}(X) \tilde{\to} \mathcal{Z}_{n-1}(X).$$

Moreover, we consider the outer differential

$$D : \mathcal{C}_{n-2}(X) \twoheadrightarrow \mathcal{B}_{n-1}(X)$$

of $(n-2)$-forms.

Lemma 2.6. *Let $\alpha, \beta \in \mathrm{VF}^{\omega}_{\mathrm{hol}}(X)$. Then $i_{[\alpha,\beta]}\omega = \mathrm{d} i_\alpha i_\beta \omega$.*

Proof. Proposition 3.1 in [8]. □

The next proposition is in some sense the version Proposition 2.5 in the volume preserving case. It is an adaption of Theorem 1 in [8].

Proposition 2.7. *Let X be a Stein manifold such that $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)$ acts transitively on X. Assume that every class of $\mathrm{H}^{n-1}(X, \mathbb{C}) = \mathcal{Z}_{n-1}(X)/\mathcal{B}_{n-1}(X)$ contains an element of $\Theta(\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)))$. Moreover, assume that there are complete vector fields $\nu_1, \ldots, \nu_N \in \mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)$ and $\mu_1, \ldots, \mu_N \in \mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)$ such that the submodule of $\mathrm{VF}_{\mathrm{hol}}(X) \wedge \mathrm{VF}_{\mathrm{hol}}(X)$ generated by the elements $\nu_j \wedge \mu_j$ is contained in the closure of $\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)) \wedge \mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X))$, and assume that there is a point $p \in X$ such that the vectors $\nu_j[p] \wedge \mu_j[p]$ are a ω-generating set for the vector space $\mathrm{T}_p X \wedge \mathrm{T}_p X$. Then X has the volume density property.*

Proof. Let $K \subset X$ be a compact set. By the identical arguments as in the proof of Proposition 2.5 we see that every element of $\mathrm{VF}_{\mathrm{hol}}(X) \wedge \mathrm{VF}_{\mathrm{hol}}(X)$ may be uniformly approximated on K by elements contained in $\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)) \wedge \mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X))$.

Let $\zeta \in \mathrm{VF}^{\omega}_{\mathrm{hol}}(X)$. By the first assumption we may, after subtracting an element of $\Theta(\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X)))$, assume that $\Theta(\zeta) \in \mathcal{B}_{n-1}(X)$. Thus $\Theta(\zeta) = D(\Psi(\gamma))$

for some $\gamma \in \mathrm{VF}_{\mathrm{hol}}(X) \wedge \mathrm{VF}_{\mathrm{hol}}(X)$. Let us approximate γ uniformly on K by elements of the form $\sum \alpha_i \wedge \beta_i \in \mathrm{Lie}(\mathrm{CVF}^\omega_{\mathrm{hol}}(X)) \wedge \mathrm{Lie}(\mathrm{CVF}^\omega_{\mathrm{hol}}(X))$ with $\alpha_i, \beta_i \in \mathrm{Lie}(\mathrm{CVF}^\omega_{\mathrm{hol}}(X))$. We use Lemma 2.6 to see that

$$D(\Psi(\sum \alpha_i \wedge \beta_i)) = \sum \mathrm{d} i_{\alpha_i} i_{\beta_i} \omega = \sum i_{[\alpha_i, \beta_i]} \omega \in \Theta(\mathrm{Lie}(\mathrm{CVF}^\omega_{\mathrm{hol}}(X))).$$

Since $\Theta(\zeta) = i_\zeta \omega$ can be approximated uniformly on K by elements of the form $\sum i_{[\alpha_i,\beta_i]}\omega = \Theta(\sum [\alpha_i, \beta_i])$ the vector field ζ can be approximated uniformly on K by elements of the form $\sum [\alpha_i, \beta_i] \in \mathrm{Lie}(\mathrm{CVF}^\omega_{\mathrm{hol}}(X))$ which proves the proposition. $\square$

2.3. **Semi-compatible and compatible pairs.** This section provides a way to find the submodules which are required in Propositions 2.5 and 2.7. The parts 2.8 - 2.11 are e.g. from [5] and [8] and are here adapted to the holomorphic case. For a vector field ν and a regular function f we denote by $\nu(f)$ the regular function which is obtained by applying ν as a derivation and $\ker \nu$ is the kernel of this linear map.

Definition 2.8. A semi-compatible pair is a pair (ν, μ) of complete vector fields such that the closure of the linear span of the product of the kernels $\ker \nu \cdot \ker \mu$ contains a non-trivial ideal $I \subset \mathcal{O}_X(X)$. We call I the ideal of (ν, μ).

Lemma 2.9. *Let (ν, μ) be a semi-compatible pair of volume preserving vector fields and I be its ideal. Then the submodule of $\mathrm{VF}_{\mathrm{hol}}(X) \wedge \mathrm{VF}_{\mathrm{hol}}(X)$ given by $I \cdot (\nu \wedge \mu)$ is contained in the closure of $\mathrm{Lie}(\mathrm{CVF}^\omega_{\mathrm{hol}}(X)) \wedge \mathrm{Lie}(\mathrm{CVF}^\omega_{\mathrm{hol}}(X))$*

Proof. Let $\tau = (\sum f_i g_i) \cdot (\nu \wedge \mu)$ with $f_i \in \ker \nu$ and $g_i \in \ker \mu$ be an arbitrary element of $\mathrm{span}\{\ker \nu \cdot \ker \mu\} \cdot (\nu \wedge \mu)$. Since $f_i \in \ker \nu$ we have $f_i \nu \in \mathrm{CVF}^\omega_{\mathrm{hol}}(X)$ for all i and similarly $g_i \nu \in \mathrm{CVF}^\omega_{\mathrm{hol}}(X)$ for all i. Thus $\tau = \sum f_i \nu \wedge g_i \mu \in \mathrm{Lie}(\mathrm{CVF}^\omega_{\mathrm{hol}}(X)) \wedge \mathrm{Lie}(\mathrm{CVF}^\omega_{\mathrm{hol}}(X))$. Therefore the closure of $\mathrm{span}\{\ker \nu \cdot \ker \mu\} \cdot (\nu \wedge \mu)$ is contained in the closure of $\mathrm{Lie}(\mathrm{CVF}^\omega_{\mathrm{hol}}(X)) \wedge \mathrm{Lie}(\mathrm{CVF}^\omega_{\mathrm{hol}}(X))$, and the claim follows. $\square$

Definition 2.10. A semi-compatible pair (ν, μ) is called a compatible pair if there is an holomorphic function $h \in \mathcal{O}_X(X)$ with $\nu(h) \in \ker \nu \setminus 0$ and $h \in \ker \mu$. Note that this condition in particular implies that $h\nu$ is a complete vector field.

Lemma 2.11. *Let (ν, μ) be a compatible pair, and let I be its ideal and h its function. Then the submodule of $\mathrm{VF}_{\mathrm{hol}}(X)$ given by $I \cdot \nu(h) \cdot \mu$ is contained in the closure of $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X))$.*

Proof. Let $f \in \ker \nu$ and $g \in \ker \mu$, then $f\nu, fh\nu, g\mu, gh\nu \in \mathrm{CVF}_{\mathrm{hol}}(X)$. A standard calculation shows

$$[f\nu, gh\mu] - [fh\nu, g\mu] = fg\nu(h)\mu \in \mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X)).$$

Thus an arbitrary element $\sum (f_i g_i)\nu(h)\mu \in \mathrm{span}\{\ker \nu \cdot \ker \mu\} \cdot \nu(h) \cdot \mu$ with $f_i \in \ker \nu$ and $g_i \in \ker \mu$ is contained in $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X))$ which concludes the proof. $\square$

Proof of Theorem 1.2. (1) Let I_i be the ideals and h_i the functions of the pairs (ν_i, μ_i) and pick any non-trivial $f_i \in I_i \cdot \nu_i(h_i)$ for every i. Since the set of points $p \in X$ where the vector fields $\mu_i[p]$ are a generating set is open and non-empty there is some $q \in X$ where the vector fields $f_i(q)\mu_i[q]$ are a generating set for $\mathrm{T}_q X$. By Lemma 2.11 the module generated by the vector fields $f_i \mu_i$ is contained in the closure of $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{hol}}(X))$ and thus by Proposition 2.5 the manifold X has the density property.

(2) Process as in *(1)*: Let I_i be the ideals of the pairs (ν_i, μ_i) and pick any non-trivial $f_i \in I_i$ for every i. Since the set of points $p \in X$ where the elements $\nu_i[p] \wedge \mu_i[p]$ are an ω-generating set is open and non-empty there is a $q \in X$ where the vector fields $f_i(q) \cdot (\nu_i[q] \wedge \mu_i[q])$ are an ω-generating set for $\mathrm{T}_q X \wedge \mathrm{T}_q X$. By Lemma 2.9 the module generated by the elements $f_i \cdot (\nu_i \wedge \mu_i)$ is contained in the closure of $\mathrm{Lie}(\mathrm{CVF}^\omega_{\mathrm{hol}}(X)) \wedge \mathrm{Lie}(\mathrm{CVF}^\omega_{\mathrm{hol}}(X))$. Thus by Proposition 2.7 the manifold X has the volume density property (the first condition of Proposition 2.7 on the cohomology group is trivially fulfilled since $\mathrm{H}^{n-1}(X, \mathbb{C}) = 0$ by assumption). $\square$

We conclude this section by two remarks. These two remarks are just for general information and are not used later in this article.

Remark 2.12. Clearly Theorem 1.2(2) still holds if we have that every class of $\mathrm{H}^{n-1}(X, \mathbb{C})$ contains an element of $\Theta(\mathrm{Lie}(\mathrm{CVF}^\omega_{\mathrm{hol}}(X)))$ as in Proposition 2.7 instead of $\mathrm{H}^{n-1}(X, \mathbb{C}) = 0$. Also, note that this condition is equivalent to the condition that every class of $\mathrm{H}^{n-1}(X, \mathbb{C})$ contains an element of $\Theta(\mathrm{CVF}^\omega_{\mathrm{hol}}(X))$. Indeed, by Lemma 2.6 all Lie brackets represent the trivial class of $\mathrm{H}^{n-1}(X, \mathbb{C})$.

Remark 2.13. There is another class of compatible pairs. Sometimes a semi-compatible pair (ν, μ) is also called compatible if there exists a function $h \in \mathcal{O}_X(X)$ with $\nu(h) \in \ker \nu \setminus 0$ and $\mu(h) \in \ker \mu \setminus 0$. For this version the identity $[f\nu, gh\mu] - [fh\nu, g\mu] = fg(\nu(h)\mu - \mu(h)\nu)$ implies that there would be a version of Theorem 1.2 where we allow compatible pairs of this kind such that the vectors $\nu(h)\mu - \mu(h)\nu$ take part in constructing the generating sets.

3. Transitivity of the $\mathrm{Aut}_{\mathrm{hol}}(X)$- and $\mathrm{Aut}^\omega_{\mathrm{hol}}(X)$-action

Let $n \geq k \geq 0$ and $a, b \in \mathbb{C}[z_0, \ldots, z_n]$ such that $\deg_{z_i}(a) \leq 2$ and $\deg_{z_i}(b) \leq 1$ for all $i \leq k$. Let $\bar{z} = (z_0, \ldots z_n)$ and $X = \{x^2 y = a(\bar{z}) + x b(\bar{z})\}$ with the holomorphic volume form $\omega = \mathrm{d}x/x^2 \wedge \mathrm{d}z_0 \wedge \ldots \wedge \mathrm{d}z_n$[1]. Consider the following vector fields on X:

$$v_x^i = \left(\frac{\partial a}{\partial z_i} + x\frac{\partial b}{\partial z_i}\right)\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z_i} \quad \text{and} \quad v_y^j = \left(\frac{\partial a}{\partial z_j} + x\frac{\partial b}{\partial z_j}\right)\frac{\partial}{\partial x} + (2xy - b(\bar{z}))\frac{\partial}{\partial z_j}$$

for $0 \leq i \leq n$ and $0 \leq j \leq k$ and moreover let

$$v_z = a(\bar{z})x\frac{\partial}{\partial x} - \left(2a(\bar{z})y - xyb(\bar{z}) + b(\bar{z})^2\right)\frac{\partial}{\partial y}.$$

Lemma 3.1. *$f(\bar{z})v_z \in \mathrm{CVF}_{\mathrm{hol}}(X)$, $f(x, z_0, ..\hat{z}_i.., z_n)v_x^i \in \mathrm{CVF}^\omega_{\mathrm{hol}}(X)$ and $f(y, z_0, ..\hat{z}_j.., z_n)v_y^j \in \mathrm{CVF}^\omega_{\mathrm{hol}}(X)$ for $f : \mathbb{C}^{n+1} \to \mathbb{C}$, $0 \leq i \leq n$ and $0 \leq j \leq k$.*

Proof. It is easy to see that the vector fields v_x^i are locally nilpotent, so they are in particular complete. The coefficients of v_y^j are linear in x and z_j for all $0 \leq j \leq k$ so the flow equation is a linear differential equation, and thus has a global solution. The vector field v_z is complete since we may first solve the linear and uncoupled differential equation for x. Then the differential equation for y becomes linear and

[1]It is a general fact that on every hypersurface $\{P = 0\} \subset \mathbb{C}^N$ there is a natural volume form given by $\omega = \left(\frac{\partial P}{\partial x_i}\right)^{-1} \mathrm{d}x_1 \wedge \ldots \wedge \widehat{\mathrm{d}x_i} \wedge \ldots \wedge \mathrm{d}x_N$.

uncoupled as well, and thus we have a global solution. It is left to show that the vector fields v_x^i and v_y^j are volume preserving. A standard calculation shows that

$$\begin{aligned} i_{v_x^i}\omega &= (-1)^{i+1}\mathrm{d}x\wedge \mathrm{d}z_0\wedge ..\widehat{\mathrm{d}z_i}..\wedge \mathrm{d}z_n, \\ i_{v_y^j}\omega &= \frac{1}{x^2}\left(\frac{\partial a}{\partial z_j}+x\frac{\partial b}{\partial z_j}\right)\mathrm{d}z_0\wedge\ldots\wedge \mathrm{d}z_n + \\ &\quad \frac{(-1)^{j+1}}{x^2}(2xy-b(\bar z))\mathrm{d}x\wedge \mathrm{d}z_0\wedge ..\widehat{\mathrm{d}z_j}..\wedge \mathrm{d}z_n \\ &= (-1)^j\ \mathrm{d}\left(\frac{a(\bar z)+xb(\bar z)}{x^2}\right)\wedge \mathrm{d}z_0\wedge ..\widehat{\mathrm{d}z_j}..\wedge \mathrm{d}z_n \\ &= (-1)^j\mathrm{d}y\wedge \mathrm{d}z_0\wedge ..\widehat{\mathrm{d}z_j}..\wedge \mathrm{d}z_n. \end{aligned}$$

Thus $L_{v_x^i}\omega=\mathrm{d}i_{v_x^i}\omega=0$ and $L_{v_y^j}\omega=\mathrm{d}i_{v_y^j}\omega=0$ which shows that they are volume preserving.

Multiplying with a kernel element doesn't affect these properties. □

Lemma 3.2. *The group* $\mathrm{Aut}_{\mathrm{hol}}(X)$ *acts transitively on* $X\setminus\{x=0\}$*. Moreover* $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)$ *acts transitively on* $X\setminus\{x=0\}$ *provided that condition (B) from the introduction holds.*

Proof. Use the flow of the vector fields v_x^i to get a transitive action on the fibers $\{x=c\neq 0\}$ and the fields v_z to connect the fibers. Thus we see that $\mathrm{Aut}_{\mathrm{hol}}(X)$ acts transitively on $X\setminus\{x=0\}$. If condition (B) holds then we may also use the vector fields v_y^j to connect the fibers. Indeed, for every $c\neq 0$ there is a point $p\in\{x=c\}$ and a vector field v_y^j such that $v_y^j[p]$ points outwards of the fiber. Thus in this case $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)$ acts transitively on $X\setminus\{x=0\}$. □

Lemma 3.3. *If (A) from the introduction holds then for every point* $p\in X\cap\{x=0\}$ *there is some* $\phi\in\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)$ *such that* $\phi(p)\notin\{x=0\}$*.*

Proof. By (A) we have that for any point $p=(0,y_0,\bar q)\in\{x=0\}\cap X$ at least one of the polynomials $b,\frac{\partial a}{\partial z_0},\ldots,\frac{\partial a}{\partial z_k}$ does not vanish at $\bar q$. For a non-vanishing $\frac{\partial a}{\partial z_j}$ the vector field v_y^j points outwards from $\{x=0\}$ at the point p. Thus the flow of v_y^j moves p away from $\{x=0\}$. If all polynomials $\frac{\partial a}{\partial z_0},\ldots,\frac{\partial a}{\partial z_k}$ vanish at $\bar q=(q_0,\ldots,q_n)$ then b is non-vanishing at $\bar q$ and moreover there is a $j\leq k$ such that $\frac{\partial a}{\partial z_j}$ does not vanish along the curve $\{z_i=q_i \text{ for all } i\neq j\}\subset\mathbb{C}^{n+1}$. This means that v_y^j is non-vanishing at p. Assume the orbit of p by the flow of v_y^j is contained in $\{x=0\}$ then the set $\{x=0,y=y_0,z_i=q_i \text{ for all } i\neq j\}\subset\mathbb{C}^{n+3}$ would contained in X and tangent to v_y^j, which is not the case. □

These two lemmas combined give the following proposition using the conditions from the introduction.

Proposition 3.4. *Assume that (A) holds. Then* $\mathrm{Aut}_{\mathrm{hol}}(X)$ *acts transitively on* X*. Assume that additionally (B) holds. Then* $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(X)$ *acts transitively on* X*.*

Corollary 3.5. *The volume preserving automorphisms act transitively on the Koras-Russell cubic* $C=\{x^2y+x+z_0^2+z_1^3=0\}$*.*

Proof. We have $\frac{\partial a}{\partial z_0}=-2z_0$ and $b(z_0,z_1)=-1$. Thus it is easy to see that (A) and (B) hold, and thus the vector fields v_x^0,v_x^1 and v_y^0 induce a transitive action on C. □

Remark 3.6. The transitivity of the action by automorphisms a priori need not to be achieved by the vector fields v_x^i, v_y^i, v_z only (which is equivalent to condition (A) from the introduction). There could be further automorphisms. For example depending on a and b there could be automorphisms of the form $(x, y, \bar{z}) \mapsto (x, \gamma y, \lambda(\bar{z}))$ where λ is an automorphisms of $\mathbb{C}^{n+1}$ with the property that $a(\lambda(\bar{z})) + xb(\lambda(\bar{z})) = \gamma(a(\bar{z}) + xb(\bar{z}))$ for some $\gamma \in \mathbb{C}^*$. A similar statement holds for transitivity by volume preserving automorphisms.

4. The Main Theorem for $n > 0$

Let $n > 0$, $n \geq k \geq 0$ and $a, b \in \mathbb{C}[z_0, \ldots, z_n]$ such that $\deg_{z_i}(a) \leq 2$ and $\deg_{z_i}(b) \leq 1$ for all $i \leq k$. Moreover, assume that not both of $\deg_{z_0}(a)$ and $\deg_{z_0}(b)$ are equal to zero. Let $\bar{z} = (z_0, \ldots z_n)$ and $X = \{x^2 y = a(\bar{z}) + xb(\bar{z})\}$ with the holomorphic volume form $\omega = \mathrm{d}x/x^2 \wedge \mathrm{d}z_0 \wedge \ldots \wedge \mathrm{d}z_n$.

Consider, again, the following vector fields on X:

$$v_x^i = \left(\frac{\partial a}{\partial z_i} + x\frac{\partial b}{\partial z_i}\right)\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z_i} \quad \text{and} \quad v_y^j = \left(\frac{\partial a}{\partial z_j} + x\frac{\partial b}{\partial z_j}\right)\frac{\partial}{\partial x} + (2xy - b(\bar{z}))\frac{\partial}{\partial z_j}$$

for $0 \leq i \leq n$ and $0 \leq j \leq k$.

Lemma 4.1. *Let $0 \leq i \leq n$ and $0 \leq j \leq k$. Then (v_x^i, v_y^j) are compatible pairs for $i \neq j$.*

First we show that (v_x^i, v_y^j) are semi-compatible pairs. Indeed the kernel of v_x^i is contained in the functions depending on $x, z_0, ..\hat{z}_i.., z_n$ and the kernel of v_y^j is contained in the functions depending on $y, z_0, ..\hat{z}_j.., z_n$ thus the closure of $\operatorname{span}\{\ker v_x^i \cdot \ker v_y^j\}$ is equal to $\mathcal{O}_X(X)$ and in particular contains an ideal.

For (v_x^i, v_y^j) being a compatible pair we need a function $h \in \ker v_y^j$ such that $v_x^i(h) \in \ker v_x^i \setminus 0$. The function $h = z_i$ does the job.

Lemma 4.2. *For a generic point $p \in X$ the vector $v_y^0[p]$ is a generating set for $\mathrm{T}_p X$ and the set $v_x^n[p] \wedge v_y^0[p]$ is a ω-generating set for $\mathrm{T}_p X \wedge \mathrm{T}_p X$. Note that the first statement also true for $n = 0$.*

Proof. Let $p = (x_0, y_0, \bar{q})$ where $\bar{q} = (q_0, \ldots, q_n)$ be such that $x_0 \neq 0$ and $\frac{\partial a(\bar{q})}{\partial z_0} + x_0 \frac{\partial b(\bar{q})}{\partial z_0} \neq 0$.

For a complete vector field $\nu \in \mathrm{CVF_{hol}}(X)$ and kernel element $f \in \ker \nu$ with $f(p) = 0$ we get an induced action (by the time one flow map of $f\nu$) on $\mathrm{T}_p X$ given by $v \mapsto v + v(f)\nu[p]$. Let $\nu_i = v_x^i$ and $f_i = x - x_0$ for $0 \leq i \leq n$. Thus the orbit of $v_y^0[p]$ under the $\mathrm{Aut}^\omega_{\mathrm{hol}}(X)_p$-action contains the vectors $v_y^0[p] + \left(\frac{\partial a(\bar{q})}{\partial z_0} + x_0 \frac{\partial b(\bar{q})}{\partial z_0}\right) v_x^i[p]$. Therefore the orbit contains a basis for $\mathrm{T}_p X$.

Since $f \in \ker v_x^n$ we have that $v_x^n[p] \mapsto v_x^n[p]$ under the actions given by $(x - x_0)v_x^i$. So in particular, similarly we have that the orbit of $v_x^n[p] \wedge v_y^0[p]$ contains a basis for $v_x^n[p] \wedge \mathrm{T}_p X$. Considering now the actions given by the vector fields $(z_n - q_n)v_x^i$ for $i \leq n-1$. We get the actions $v_x^n[p] \mapsto v_x^n[p] + x_0^2 v_x^i[p]$, and thus we see that the orbit of the $\mathrm{Aut}^\omega_{\mathrm{hol}}(X)_p$-action contains a basis for $v_x^i[p] \wedge \mathrm{T}_p X$ for all $i \leq n$. Together they build then together a basis for $\mathrm{T}_p X \wedge \mathrm{T}_p X$. □

Proof of the Main Theorem for $n > 0$. By Lemma 4.1 there exists a point $p \in X$ and compatible pairs (ν_i, μ_i) such that the vectors $\mu_i[p]$ are a generating set for $\mathrm{T}_p X$ and the elements $\nu_i[p] \wedge \mu_i[p]$ are a ω-generating set for $\mathrm{T}_p X \wedge \mathrm{T}_p X$ thus by Theorem 1.2 the claim is proven. Proposition 3.4 proves the "in particular" part. □

Remark 4.3. We never used that a and b are polynomials. In fact, the Main Theorem also holds if a and b are polynomial in z_0 and analytic in $z_1, \dots, z_n$.

Remark 4.4. The condition that $\mathrm{H}^{n+1}(X, \mathbb{C}) = 0$ in the Main Theorem could be omitted in the case when every class of $\mathrm{H}^{n+1}(X, \mathbb{C})$ contains an element of $\Theta(\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{hol}}(X))$ as in Proposition 2.7. However this cannot be achieved by the complete vector fields $f \cdot v_x^i$ and $g \cdot v_y^j$ since they are all mapped to the zero class by Θ (see the calculation in the proof of Lemma 3.1). So the existence of other complete volume preserving vector fields would be required.

Remark 4.5. If $a(\bar{z})$ is reduced (i.e. has connected fibers) and $\{a(\bar{z}) = 0\} \subset \mathbb{C}^{n+1}$ is smooth then the condition $\mathrm{H}^{n+1}(X, \mathbb{C})$ is actually equivalent to the condition that $\tilde{\mathrm{H}}^{n-1}(\{a(\bar{z}) = 0\}, \mathbb{C}) = 0$. Indeed, if $Y = \{uv = a(\bar{w})\}$ then the map $X \to Y$ given by $(x, y, \bar{z}) \mapsto (x, xy - b(\bar{z}), \bar{z})$ is an affine modification in the sense of [9]. Theorem 3.1 of [9] shows that $\mathrm{H}^*(X, \mathbb{C}) = \mathrm{H}^*(Y, \mathbb{C})$. Moreover, Proposition 4.1 of [9] states that we have

$$\tilde{\mathrm{H}}^*(Y, \mathbb{C}) = \tilde{\mathrm{H}}^{*-2}\Big(\{a(\bar{z}) = 0\}, \mathbb{C}\Big)$$

which proofs the statement of the remark.

5. The Main Theorem for $n = 0$

Let $a, b \in \mathbb{C}$, and let $X_{a,b} = \{x^2 y = z^2 - b + ax\}$. Note that $X_{a,b}$ is smooth if and only if a and b are not both equal to zero. Also, the conditions (A) and (B) from the introduction hold automatically if $X_{a,b}$ is smooth. Therefore the volume preserving automorphisms act transitively on $X_{a,b}$. The following proposition is mostly taken from [14], and among others it shows that $X_{a,b}$ is algebraically isomorphic to $X_{1,0}$, $X_{0,1}$ or $X_{1,1}$. Moreover it shows that $X_{0,1}$ and $X_{1,1}$ are biholomorphic.

Proposition 5.1. *(a) Let $a \in \mathbb{C}$ and $b \in \mathbb{C}^*$ then*

- *(i) $X_{a,1} \cong_{\mathrm{alg}} X_{a,b}$ and $X_{b,a} \cong_{\mathrm{alg}} X_{1,a}$,*
- *(ii) $X_{0,1} \cong_{\mathrm{hol}} X_{a,1}$,*

where $\cong_{\mathrm{alg}}$ means isomorphic as algebraic surfaces and $\cong_{\mathrm{hol}}$ isomorphic as complex manifolds.

(b) Let $X = \{x^2 y = p(x, z)\}$ be a smooth hypersurface with $p \in \mathbb{C}[x, z]$ and $\deg_z p(0, z) \le 2$ then

- *(i) if $\deg_z p(0, z) = 0$ then $X \cong_{\mathrm{alg}} \mathbb{C}^* \times \mathbb{C}$,*
- *(ii) if $\deg_z p(0, z) = 1$ then $X \cong_{\mathrm{alg}} \mathbb{C}^2$,*
- *(iii) if $p(0, z)$ has a zero with multiplicity 2 then $X \cong_{\mathrm{alg}} X_{1,0}$,*
- *(iv) if $p(0, z)$ has two different zeroes then there is a unique $a \in \{0, 1\}$ such that $X \cong_{\mathrm{alg}} X_{a,1}$.*

Proof. Theorem 9 from [14] gives the isomorphisms in (a)(i). The biholomorphic map in (a)(ii) is given by

$$(x, y, z) \mapsto \left(x, e^{-ax} y + \frac{e^{-ax} + ax - 1}{x^2}, e^{-\frac{a}{2}x} z \right).$$

Theorem 5 from [14] states that X is algebraically isomorphic to $x^2 y = s(z) + x t(z)$ for some $s, t \in \mathbb{C}[z]$ with $\deg t \le \deg s - 2$ Following the given algorithm we see that $s(z) = p(0, z)$ which is then, after a linear change in the z-coordinate, given by (i) 1, (ii) z, (iii) z^2 or (iv) $z^2 - 1$. The isomorphisms in (b) are then easily found and the uniqueness in (b)(iv) follows from Theorem 9 from [14]. □

Despite of Proposition 5.1(a) we will start working on $X_{a,b}$ for general $a, b \in \mathbb{C}$. It turns out that this is more convenient for most arguments.

Remark 5.2. Every function $f \in \mathbb{C}[X_{a,b}]$ can be written uniquely as

$$f(x,y,z) = \sum_{i=1}^{\infty} x^i a_i(z) + \sum_{i=1}^{\infty} xy^i b_i(z) + \sum_{i=1}^{\infty} y^i c_i(z) + d(z),$$

indeed replace every x^2y by $z^2 - b + ax$. Alternatively f can be written uniquely as

$$f(x,y,z) = f_1(x,y) + z f_2(x,y),$$

indeed replace every z^2 by $x^2y + b - ax$.

We will use a tool in order to proof the (volume) density property, namely the algebraic (volume) density property. Note that the algebraic (volume) density property implies the (volume) density property, see [6].

Definition 5.3. Let X be an affine algebraic manifold. If the Lie algebra $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{alg}}(X))$ generated by complete algebraic vector fields $\mathrm{CVF}_{\mathrm{alg}}(X)$ on X is equal to the Lie algebra of all algebraic vector fields $\mathrm{VF}_{\mathrm{alg}}(X)$ on X then X has the algebraic density property.

Let X be an affine algebraic manifold equipped with an algebraic volume form ω. If the Lie algebra $\mathrm{Lie}(\mathrm{CVF}^{\omega}_{\mathrm{alg}}(X))$ generated by complete volume preserving algebraic vector fields $\mathrm{CVF}^{\omega}_{\mathrm{alg}}(X)$ on X is equal to the Lie algebra of all volume preserving algebraic vector fields $\mathrm{VF}^{\omega}_{\mathrm{hol}}(X)$ on X then X has the algebraic volume density property.

5.1. **The volume density property.** Let $a, b \in \mathbb{C}$ such that not both equal to zero. For proving the volume density property for $X_{a,b} = \{x^2y = z^2 - b + ax\}$ with respect to $\omega = \mathrm{d}x/x^2 \wedge \mathrm{d}z$ we will need the following two vector fields:

$$v_x = 2z\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z}, \quad v_y = 2z\frac{\partial}{\partial x} + (2xy - a)\frac{\partial}{\partial z}.$$

Lemma 3.1 translates into:

Lemma 5.4. *$x^k v_x \in \mathrm{CVF}^{\omega}_{\mathrm{alg}}(X_{a,b})$ and $y^k v_y \in \mathrm{CVF}^{\omega}_{\mathrm{alg}}(X_{a,b})$ for $k \geq 0$.*

Lemma 5.5. *Let $v \in \mathrm{VF}^{\omega}_{\mathrm{alg}}(X_{a,b})$. Then the 1-form $i_v\omega$ is exact and $i_v\omega = \mathrm{d}f$ defines a bijection between algebraic volume preserving vector fields and algebraic functions modulo constants.*

The functions corresponding to $x^k v_x$ and $y^k v_y$ are given by the equations

$$(k+1) i_{(x^k v_x)}\omega = -\mathrm{d}x^{k+1} \text{ and } (k+1) i_{(y^k v_k)}\omega = \mathrm{d}y^{k+1}.$$

Proof. The correspondence is given by the isomorphism Θ the map D in Section 2 using the fact that $\mathrm{H}^1(X_{a,b}, \mathbb{C}) = 0$. The same correspondence was also used in [12]. The triviality of $\mathrm{H}^1(X_{a,b}, \mathbb{C})$ follows from the fact that $X_{a,b}$ is an affine modification of $\mathbb{C}^2$ along the divisor $2 \cdot \{x = 0\}$ with center at the ideal $(x^2, z^2 - b + ax)$ using the notation from [9]. If $b \neq 0$ then Proposition 3.1 from [9] shows that $X_{a,b}$ is simply connected (since $\mathbb{C}^2$ is simply connected). If $b = 0$ then Theorem 3.1 from

[9] shows in a similar way that $\mathrm{H}^1(X_{a,0},\mathbb{C}) = \mathrm{H}^1(\mathbb{C}^2,\mathbb{C})$, and thus is trivial. For the two identities we make the calculations:

$$(k+1)i_{(x^k v_x)}\omega = (k+1)x^k i_{v_x}\omega = -(k+1)x^k \mathrm{d}x = -\mathrm{d}x^{k+1},$$

$$(k+1)i_{(y^k v_y)}\omega = (k+1)y^k i_{v_y}\omega = (k+1)y^k \left(\frac{2z}{x^2}\mathrm{d}z - \frac{2xy-a}{x^2}\mathrm{d}x\right) =$$

$$= (k+1)y^k \mathrm{d}\frac{z^2-b+ax}{x^2} = (k+1)y^k \mathrm{d}y = \mathrm{d}y^{k+1}.$$

□

Recall that for a vector field ν and a regular function f we denote by $\nu(f)$ the regular function which is obtained by applying ν as a derivation.

Lemma 5.6. *Let $v_1, v_2 \in \mathrm{VF}^\omega_{\mathrm{alg}}(X_{a,b})$ and $i_{v_1}\omega = \mathrm{d}f$ then $i_{[v_2,v_1]}\omega = \mathrm{d}v_2(f)$. In particular, if f corresponds to a vector field in $\mathrm{Lie}(\mathrm{CVF}^\omega_{\mathrm{alg}}(X_{a,b}))$ then $x^k v_x(f)$ and $y^k v_y(f)$ correspond to a vector field in $\mathrm{Lie}(\mathrm{CVF}^\omega_{\mathrm{alg}}(X_{a,b}))$.*

Proof. The identity $i_{[v_2,v_1]}\omega = \mathrm{d}v_2(f)$ is shown in Lemma 3.2 in [12]. □

Lemma 5.7. *Let $i,j,k \geq 0$. Then*

$$v_x(y^{j+1}) = 2(j+1)y^j z, \tag{1}$$

$$v_x(y^{j+1}z^{k+1}) = y^j z^k(2(j+1)z^2 + (k+1)(z^2-b+ax)), \tag{2}$$

$$y^j v_y(z^{k+1}) = (k+1)y^j z^k(2xy-a), \tag{3}$$

$$v_y(x^{i+1}) = 2(i+1)x^i z, \tag{4}$$

$$v_y(x^{i+1}z^{k+1}) = x^i z^k\left(2(i+1)z^2 + (k+1)(2z^2-2b+ax)\right). \tag{5}$$

Proof. The lemma is proven by straight forward calculations. □

Proposition 5.8. *Smooth surfaces $X_{a,b} = \{x^2y = z^2 - b + ax\}$ have the algebraic volume density property with respect to $\omega = \mathrm{d}x/x^2 \wedge \mathrm{d}z$.*

Proof. Let L be the set of function that corresponds to the Lie algebra of complete vector fields. By Lemma 5.5 we already have $x^i \in L$ and $y^i \in L$ for $i \geq 0$. We need to show that all functions on X (modulo constants) are contained in L. It is enough to show that (a) $x^i z^{k+1} \in L$, (b) $xy^{j+1}z^k \in L$ and (c) $y^{j+1}z^{k+1} \in L$ for all $i,j,k \geq 0$.

First we show (a) $x^i z^{k+1} \in L$: The statement (a) is also true for $k = -1$ by Lemma 5.5. Lemma 5.6 shows that $v_y(x^{i+1}) \in L$. Therefore by (4) we get $2(i+1)x^i z \in L$ and thus $x^i z \in L$ for $i \geq 0$ which is the statement for $k = 0$. Let us assume that the statement is true for $k-1$ and for k. Then, by Lemma 5.6 we have $v_y(x^{i+1}z^k) \in L$. By the induction assumption and (5) we have also $x^i z^{k+1} \in L$ which concludes the proof of (a) $x^i z^{k+1} \in L$ inductively for all $i,k \geq 0$.

The next step is to show (b) $xy^{j+1}z^k \in L$ and (c) $y^{j+1}z^{k+1}$ for $k = 0$: Note that (c) holds also for $k = -1$ by Lemma 5.5. By Lemma 5.6 we have $v_x(y^{j+2}) \in L$, and thus by (1) $y^{j+1}z \in L$ which proofs statement (c) for $j \geq 0$ and $k = 0$. By the same lemma we have $y^j v_y(z) \in L$. Thus by (3) and (c) we have $xy^{j+1} \in L$ proving statement (b) for $j \geq 0$ and $k = 0$.

The last step is to show (b) $xy^{j+1}z^k \in L$ and (c) $y^{j+1}z^{k+1}$ for arbitrary k: Let us assume that (b) and (c) hold for k and moreover (c) holds for $k-1$. By Lemma 5.6 and the induction assumption we have $v_x(y^{j+1}z^{k+1}) \in L$ for all $j \geq 0$. Thus by the induction assumption and (2) we also have $y^j z^{k+2} \in L$ for all $j \geq 0$ which is statement (c) for $k+1$. Similarly, we have $y^j v_y(z^{k+2}) \in L$, and thus by (3) and the induction assumption we get $xy^{j+1}z^{k+1} \in L$ for all $j \geq 0$. This is statement (b) for $k+1$. Thus by induction over k the statements (b) and (c) are shown. $\square$

Theorem 5.9. *Let $X = \{x^2y = p(x,z)\}$ with $p \in \mathbb{C}[x,z]$ and $\deg(p(0,z)) \leq 0$ be a smooth surface, then X has the algebraic volume density property for the volume form $\mathrm{d}x/x^2 \wedge dz$.*

Proof. If $\deg(p(0,z)) \in \{1,2\}$ then by Proposition 5.1 the surface X is algebraically isomorphic to some surface $X_{a,b}$ or to $\mathbb{C}^2$. Since on these surfaces there are no non-constant invertible regular functions two different algebraic volume forms differ only by multiplication with a constant. So the isomorphism induces a natural bijection between algebraic volume preserving vector fields on X and $X_{a,b}$ (resp. $\mathbb{C}^2$). Thus algebraic volume density property is preserved under algebraic isomorphisms. Therefore Proposition 5.8 and the well known fact that $\mathbb{C}^2$ has the algebraic volume density property concludes this case.

If $\deg(p(0,z)) = 0$ then by Proposition 5.1 the surface X is algebraically isomorphic to $\mathbb{C}^* \times \mathbb{C}$. For any algebraic volume form η on $\mathbb{C}^* \times \mathbb{C}$ there is an algebraic automorphism such that the pull back of η is equal to $\eta_0 = \mathrm{d}u/u \wedge \mathrm{d}v$. Indeed for an arbitrary $\eta = au^k \mathrm{d}u \wedge \mathrm{d}v$ with $a \in \mathbb{C}^*$ and $k \in \mathbb{Z}$ the pull back of η by $(u,v) \mapsto (u, a^{-1}u^{-k-1}v)$ is η_0. Apply Theorem 1 of [8] to the semi-compatible pair $(u \cdot \partial/\partial u, \partial/\partial v)$ to see that $\mathbb{C}^* \times \mathbb{C}$ with η_0 has the algebraic volume density property and thus $\mathbb{C}^* \times \mathbb{C}$ has the algebraic volume density property for all algebraic volume forms. $\square$

5.2. **The density property.** We will show the density property for the surfaces $X_{1,b} = \{x^2y = z^2 - b + x\}$ with $b \in \mathbb{C}$. We will use the following vector fields on $X_{1,b}$:

$$v_x = 2z\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z}, \quad v_y = 2z\frac{\partial}{\partial x} + (2xy-1)\frac{\partial}{\partial z},$$

$$v_z = z^2x\frac{\partial}{\partial x} - \left(2(z^2-b)y - axy + a^2\right)\frac{\partial}{\partial y}.$$

Definition 5.10. For an algebraic vector field ν the ω-divergence $\mathrm{div}_\omega \nu$ is the regular function given by $\mathrm{d}i_\nu\omega = (\mathrm{div}_\omega\nu)\cdot\omega$.

Lemma 5.11. *We have $x^kv_x, y^kv_y \in \mathrm{CVF}^\omega_{\mathrm{alg}}(X_{1,b})$ and $zx^kv_x, z^kv_z \in \mathrm{CVF}_{\mathrm{alg}}(X_{1,b})$ for $k \geq 0$. Moreover $\mathrm{div}_\omega zx^kv_x = x^{k+2}$ and $\mathrm{div}_\omega z^kv_z = -z^{k+2}$.*

Proof. Lemma 5.4 says $x^kv_x, y^kv_y \in \mathrm{CVF}^\omega_{\mathrm{alg}}(X_0)$ and $zx^kv_x \in \mathrm{CVF}_{\mathrm{alg}}(X_{1,b})$. For the statement about the divergence we make the calculations

$$\mathrm{d}i_{zx^kv_x}\omega = -\mathrm{d}zx^k\mathrm{d}x = x^k\mathrm{d}x \wedge \mathrm{d}z = x^{k+2}\omega,$$

$$\mathrm{d}i_{z^kv_z}\omega = \mathrm{d}\frac{z^{k+2}}{x}\mathrm{d}z = \frac{-z^{k+2}}{x^2}\mathrm{d}x \wedge \mathrm{d}z = -z^{k+2}\omega,$$

and thus prove the statement. $\square$

The following lemma is well known.

Lemma 5.12. *For $v_1, v_2 \in \mathrm{VF}_{\mathrm{alg}}(X_{1,b})$ we have*

$$\mathrm{div}_\omega[v_1, v_2] = v_1(\mathrm{div}_\omega v_2) - v_2(\mathrm{div}_\omega v_1),$$

in particular we have

$$\mathrm{div}_\omega[x^k v_x, v_2] = x^k v_x(\mathrm{div}_\omega v_2) \quad \text{and} \quad \mathrm{div}_\omega[y^k v_y, v_2] = y^k v_y(\mathrm{div}_\omega v_2)$$

for any $k \geq 0$. Moreover for $f \in \mathbb{C}[X_{1,b}]$:

$$\mathrm{div}_\omega f v = f \mathrm{div}_\omega v + v(f).$$

Lemma 5.13. *Let $f \in \mathbb{C}[X_{1,b}]$. Then $fv_y \in \mathrm{Lie}(\mathrm{CVF}_{\mathrm{alg}}(X_{1,b}))$.*

Proof. Let $E \subset \mathbb{C}[X_{1,b}]$ be given by $E = \{\mathrm{div}_\omega \nu : \nu \in \mathrm{Lie}(\mathrm{CVF}_{\mathrm{alg}}(X_{1,b}))\}$. It is enough to show that for every $f \in \mathbb{C}[X_{1,b}]$ we have $\mathrm{div}_\omega(fv_y) \in E$. Indeed, then $fv_y - \nu \in \mathrm{VF}^\omega_{\mathrm{alg}}(X_{1,b})$ for some $\nu \in \mathrm{Lie}(\mathrm{CVF}_{\mathrm{alg}}(X_{1,b}))$. Thus by Proposition 5.8 we have $fv_y \in \mathrm{Lie}(\mathrm{CVF}_{\mathrm{alg}}(X_{1,b}))$.

By Lemma 5.11 we have $x^{i+2} \in E$ and $z^{i+2} \in E$ for all $i \geq 0$. Thus by Lemma 5.12 we get $v_y(x^{i+2}) = 2(i+2)x^{i+1}z \in E$, and therefore $v_y(xz) = 2z^2 + 2x^2y - x = 4z^2 - 2b + x \in E$. Since $z^2 \in E$ we get $x - 2b \in E$, and thus $v_y(x - 2b) = 2z \in E$. Altogether we have $x^i z \in E$, $x^{i+2} \in E$ and $x - 2b \in E$ for all $i \geq 0$.

Let $f = x^i y^j z^k$ for $i, j \geq 0$ and $k \in \{0, 1\}$. If $f \neq xy^j$ then we have

$$\mathrm{div}_\omega(fv_y) = y^j v_y(x^i z^k) \in E$$

by Lemma 5.12 because $x^i z^k \in E$ by the above. If $f = xy^j$ then

$$\mathrm{div}_\omega(fv_y) = y^j v_y(x) = y^j v_y(x - 2b) \in E$$

by the same arguments. Thus the lemma is proven since any regular function is a sum of such monomials by Remark 5.2. □

Proposition 5.14. *The surface $X_{1,b}$ has the density property.*

Proof. By Lemma 4.2 the tangent vectors of v_y are a generating set for the tangent space $\mathrm{T}_q X_{1,b}$ at a generic point $q \in X_{1,b}$. By Lemma 5.13 the $\mathbb{C}[X_{1,b}]$- submodule generated by v_y is contained in $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{alg}}(X_{1,b}))$. Thus the $\mathcal{O}_{X_{1,b}}(X_{1,b})$- submodule generated by v_y is contained in the closure of $\mathrm{Lie}(\mathrm{CVF}_{\mathrm{alg}}(X_{1,b}))$. Therefore by Lemma 4.2 and Proposition 2.5 the surface $X_{1,b}$ has the density property. □

Theorem 5.15. *Let $X = \{x^2y = p(x, z)\}$ with $p \in \mathbb{C}[x, z]$ and $\deg(p(0, z)) \leq 2$ be a smooth surface. Then X has the density property.*

Proof. By Proposition 5.1 the surface X is biholomorphic to $\mathbb{C}^* \times \mathbb{C}$, $\mathbb{C}^2$, $X_{1,0}$ or $X_{1,1}$. It is well known that the first two have the density property. The two other surfaces have the density property by Proposition 5.14. □

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